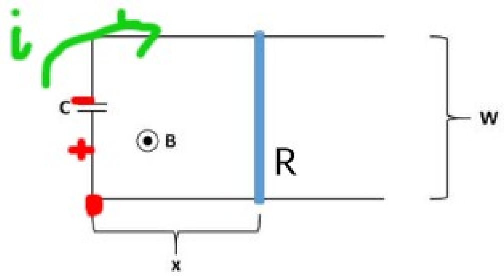
A rod with resistance R is placed on two parallel, horizontal rails, with no resistance, that are a distance W apart. The rails are connected at one end with a capacitor C. At time $t = 0$, an external magnetic field is turned on that points out of the page, with magnitude $|\vec{B}| = B_0$. The rod starts at a distance x from the capacitor and is moved to the right with constant velocity v_0.



Start by writing Faraday's law of induction. Technically there should be a minus sign here, but it's easier to just use the equation to calculate the magnitude of the induced emf and to use Lenz's law to determine the direction.

$$\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = \mathcal{E}$$

Rewriting this in terms of the magnetic field B_0 and area $A = xW$ gives

$$\mathcal{E} = \frac{\mathrm{d}}{\mathrm{d}t} B_0 x W = B_0 v W$$

Also, $\mathcal{E} = iR$, so

$$i_{\text{induced}} = \frac{B_0 v W}{R}$$

Since the flux out of the page increases as the area of the loop increases, the induced current will tend to cancel out this increase. This means that the induced current must flow clockwise so as to produce a field pointing *into* the page.

Now we apply the Kirchoff loop rule to the circuit.

$$\boxed{-\frac{q}{C} - \frac{\mathrm{d}q}{\mathrm{d}t} R + BvW = 0}$$

This means that there is a constant clockwise emf BvW produced by the slide wire, but it is counteracted both by the resistance of the circuit.

b) Solve the equation from part a) to find the charge on the capacitor as a function of time.

This is a linear first-order non-homogenous differential equation. We can solve it by means of an integrating factor. We begin by rearranging the equation so that the coefficient of $\frac{\mathrm{d}q}{\mathrm{d}t}$ is 1.

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{RC} q = \frac{BvW}{R}$$

Now we multiply both sides by the integrating factor which is e raised to the integral of the coefficient of q(t)

$$u = e^{\int \frac{1}{RC} \mathrm{d}t}$$

This gives

$$\frac{\mathrm{d}q}{\mathrm{d}t} e^{t/RC} + q \cdot \frac{1}{RC} e^{-t/RC} = \frac{BvW}{R} e^{t/RC}$$

The left hand side is now recognizable as the derivative of a product of two functions

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(uq\right) = \frac{BvW}{R} e^{t/RC}$$

Integrating both sides and dividing by u gives

$$q(t) = \frac{BvW}{R} e^{-t/RC} \int e^{t/RC} \mathrm{d}t$$

$$q(t) = \frac{BvW}{R} e^{-t/RC} \left[RCe^{t/RC} + D\right] = BvWC + \frac{BvWD}{R} e^{-t/RC}$$

(where D is some unknown constant)

Plugging in the initial condition $q(0) = 0$ gives

$$q(0) = BvWC + \frac{BvWD}{R} = 0,$$

so $D = -RC$, and finally by multiplying everything together, we have

$$\boxed{q(t) = BvWC\left(1 - e^{-t/RC}\right)}$$

This makes sense because it satisfies the initial conditions and because as $t \to \infty$, the charge approaches a constant value which is equal to BvW (the emf due to the sliding wire) multiplied by the capacitance.